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ON QUASIPERIODIC BOUNDARY VALUE PROBLEMS AND THEIR APPLICATIONS IN THE THEORY OF ELASTICITY*

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The fundamental boundary value problems of analytic function theory are considered on certain systems of contours possessing translational symmetry: the Riemann problem on an oblique lattice of arbitrary contours, the Hilbert problem and the mixed problem for a half-plane, the Dirichlet problem for a plane with a periodic system of slits on a line. In contrast to /1/, where the listed problems are solved under the condition of periodicity of their coefficients and free terms, this condition is here imposed only on the coefficients. By applying a discrete Fourier transform and periodicity of the boundary conditions for an elementary cell, the formulated quasiperiodic problems are reduced to periodic problems and are solved in closed form. The results obtained are used to solve (in quadratures) new mixed problems of elasticity theory in translationally symmetric domains with nonperiodic boundary conditions.

1. The Riemann problem. In the plane of the complex variable $z = x + iy$ let a system of smooth contours L_k, $k = 0, \pm 1, \ldots$ be given that possesses one-dimensional translational symmetry with the basis vector $\omega = \omega_1 + i\omega_2$, the motion of the plane a quantity ω transforms L_k into L_{k+1}. The strip $|\operatorname{Re}(ze^{-i\xi})| \leqslant {}^1/_2 |\omega|$ of width $|\omega|$ with slope $\xi = \arg \omega$, measured counterclockwise from the x axis is selected as the elementary cell Ω_0 and $L_0 \subset \Omega_0$. Find the solution of the Riemann boundary value problem /1/

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t); \quad t \in L, \quad L = \bigcup_{k=-\infty}^{\infty} L_k \tag{1.1}$$

for a piecewise-analytic function $\Phi(z)$ decreasing at infinity under the assumption that

$$G(t+\omega) = G(t), \; G(t) \neq 0, \; t \in L_0 \tag{1.2}$$

and that the functions $G(t)$ and $g(t)$ satisfy the Hölder conditions H_1 and H_*, respectively. The conditions H_M and H_* imposed on the function $f(t)$ are understood to be the conditions (M is an integer)

$$H_M: f(t + M\omega) = f(t), \; |f(t_1) - f(t_2)| < A\, |t_1 - t_2|^\lambda$$
$$t_1, t_2 \in \bigcup_{k=0}^{M-1} L_k, \quad 0 < \lambda \leqslant 1$$
$$H_*: |f(t_1) - f(t_2)| < A_k |t_1 - t_2|^\lambda; \quad t_1, t_2 \in L_k, \quad k = 0, \pm 1, \ldots$$
$$\sum_{k=-\infty}^{\infty} A_k < A$$

Problems of this kind with the periodic coefficient $G(t)$ and the arbitrary free term $g(t)$ will be called quasiperiodic.

We introduce the discrete Fourier transform of the function $f(z)$ in the form

$$f_*(z + s\omega, \varphi) = \sum_{k=-\infty}^{\infty} f(z + s\omega + k\omega)e^{-ik\varphi}, \quad z \in \Omega_0 \tag{1.3}$$
$$s = 0, \pm 1, \ldots, \varphi \in [0, 2\pi]$$

According to the theory of Fourier series, the inversion formula

$$f(z + s\omega + k\omega) = \frac{1}{2\pi}\int_0^{2\pi} f_*(z + s\omega, \varphi)e^{ik\varphi}\, d\varphi;$$
$$z \in \Omega_0, \quad k = 0, \pm 1, \ldots$$

*Prikl.Matem.Mekhan.,46,No.5,821-830,1982 holds at those points z where the series (1.3) converges uniformly with respect to φ in the interval $[0, 2\pi]$.

Hence, setting $k = 0$, we obtain

$$f(z + s\omega) = \frac{1}{2\pi}\int_0^{2\pi} f_*(z + s\omega, \varphi)\, d\varphi, \quad z \in \Omega_0 \tag{1.4}$$

The identity

$$f_*(z + s\omega, \varphi) = e^{is\varphi} f_*(z), \quad z \in \Omega_0, \quad s = 0, \pm 1, \ldots \tag{1.5}$$

follows from (1.3).

We assume that the sum of the series (1.3), comprised of piecewise-analytic functions $f(z) = \Phi(z)$ is piecewise-analytic, and its limit values on L_0 satisfy the relationship (1.1) for the transform. Then because of applying the transform (1.3) to the problem (1.1), the following boundary value problem for the strip Ω_0 occurs in conformity with (1.2) (the parameter φ is omitted):

$$\begin{gathered}\Phi_*^+(t) = G(t)\Phi_*^-(t) + g_*(t), \quad t \in L_0 \\ \Phi_*(z + \omega) = e^{i\varphi}\Phi_*(z); \quad z \in \{\operatorname{Re}(ze^{-i\xi}) = -{}^1/_2|\omega|\}\end{gathered} \tag{1.6}$$

where the functions $G(t)$ and $g_*(t)$ satisfy the condition H_1 on L_0. The second condition of (1.6) is obtained on the boundary of the strip Ω_0 from the condition of continuity of the function $\Phi(z)$ on the total boundary of the adjacent strips Ω_k and Ω_{k+1}, $k = 0, \pm 1, \ldots$ by using (1.5). *It shows that the problem (1.6) is not periodic in the plane. However, this problem* can be made periodic by making the substitution /2/

$$\Phi_*(z) = e^{\alpha z}\,\Phi_0(z), \quad \alpha = i\varphi\omega^{-1} \tag{1.7}$$

The solution of the appropriate periodic problem

$$\begin{gathered}\Phi_0^+(t) = G(t)\,\Phi_0^-(t) + e^{-\alpha t}\, g_*(t), \quad t \in L_0 \\ \Phi_0(z + \omega) = \Phi_0(z), \quad z \in \{\operatorname{Re}(ze^{-i\xi}) = -{}^1/_2\,|\,\omega\,|\}\end{gathered} \tag{1.8}$$

is constructed by the method of F.D. Gakhov. Let $L_k = L_{k1} \cup L_{k2} \ldots \cup L_{kN}$, where L_{0n} is a simple *open or closed contour,* $\varkappa_n$ *is the index of the coefficient* $G(t)$ *in* L_{0n} *in a given class of* piecewise-analytic automorphic functions $\Phi_0(z)$, $P_\varkappa(t)$ are polynomials of degree $\varkappa$ with the coefficients $C_n = C_n(\varphi)$. We then have (/1/, Sect.43,52)

$$\begin{gathered}\Phi_0(z) = X(z)\left[\int_{L_0} \frac{e^{\beta t - \alpha t} g_*(t)\, dt}{\omega(e^{\beta t} - e^{\beta z})\, X^+(t)} + P_{\varkappa-1}(e^{\beta z})\right] \\ X(z) = e^{\Gamma(z)} \prod_{n=1}^{N} (e^{\beta z} - e^{\beta b_n})^{-\varkappa_n}; \quad \Gamma(z) = \frac{1}{\omega}\int_{L_0} \frac{\ln G(t)\, e^{\beta t}\, dt}{e^{\beta t} - e^{\beta z}} \\ P_\varkappa(t) = \sum_{n=1}^{\varkappa+1} C_n t^{n-1}, \quad \varkappa = \sum_{n=1}^{N} \varkappa_n, \quad \beta = \frac{2\pi i}{\omega}\end{gathered} \tag{1.9}$$

where b_n is a finite point of the open and an arbitrary point of the closed contour L_{0n}, and the chosen branch of the function $X(z)$ corresponds to the condition $\lim w^\varkappa X(w) = 1$, $w = e^{\beta z} \to \infty$.

Now, the solution of the problem (1.1) in the class of piecewise-analytic functions decreasing at infinity can be found from (1.4), (1.7), (1.9). Taking the periodicity of the function $\Phi_0(z)$ into account, we obtain $(z \in \Omega_0)$

$$\Phi(z + s\omega) = \frac{X(z)}{2\pi}\int_0^{2\pi}\left[\int_{L_0} \frac{\exp(\beta t - \alpha t + \alpha z + i\varphi s)\, g_*(t)\, dt}{\omega(e^{\beta t} - e^{\beta z})\, X^+(t)} + e^{\alpha z + i\varphi s} P_{\varkappa-1}(w)\right] d\varphi \tag{1.10}$$

If commutation of the order of integration in t and φ is allowable, this expression simplifies to

$$\Phi(z + s\omega) = \frac{X(z)}{2\pi i}\left\{\int_{L_0}\left[\sum_{k=-\infty}^{\infty} \frac{g(t + k\omega)}{t - z + (k - s)\omega}\right]\frac{dt}{X^+(t)} + i\int_0^{2\pi} e^{\alpha z + i\varphi s} P_{\varkappa-1}(e^{\beta z})\, d\varphi\right\} \tag{1.11}$$

Here the transform $g_*(t)$, written explicitly in the form of a series in k, is integrated with respect to φ. If this series diverges, then by weakening the requirement on the function $g(t)$ and changing the course of the solution, it is expedient to interchange integration with respect to t and summation with respect to k; the summation sign with respect to k can always be extracted from under the integral sign by the same means.

Indeed, let the function $g(t)$ satisfy the condition H_* without constraint on A_k, and let it have algebraic growth with respect to t, for example. Then condition H_* is satisfied in the problem (1.1) with the free term $g_s(t) = g(t)\delta_{ks}$, $t \in L_k$, where δ_{ks} is the Kronecker delta. The appropriate solution (1.10) can be called the Green's s-function of the initial problem (1.1). Summing the s-functions with respect to s between $-\infty$ and ∞, we obtain the solution of the problem (1.1) in the form (1.10) with interchanged summation and integration signs. The procedure mentioned is applicable, in particular, when the free term is a periodic function of the form $g(t + M\omega) = g(t)$ that satisfies the condition H_M. However, in this case there is a simple solution based on replacing the discrete transformation (1.3)—(1.5) by the finite transformation

$$f_*(z + s\omega) = \sum_{k=0}^{M-1} f(z + s\omega + k\omega)e^{-ik\varphi}, \quad \varphi = \frac{2\pi m}{M} \tag{1.12}$$

$$f(z + s\omega + k\omega) = \frac{1}{M}\sum_{m=0}^{M-1} f_*(z + s\omega)e^{ik\varphi}$$

$$f_*(z + s\omega) = e^{is\varphi}f_*(z); \quad z \in \Omega_0, \quad m, s = 0, 1, \ldots M-1$$

Repeating the previous discussion, it can be confirmed that this solution is analogous to (1.10) and has the form

$$\Phi(z + s\omega) = \frac{X(z)}{M}\sum_{m=0}^{M-1}\left\{\int_{L_0} \frac{\exp(\beta t - \alpha t + \alpha z + i\varphi s)g_*(t)\,dt}{\omega(e^{\beta t} - e^{\beta z})X^+(t)} + e^{\alpha z + i\varphi s}P_{\varkappa-1}(e^{\beta z})\right\} \tag{1.13}$$

The quantities φ and $g_*(t)$ are evaluated by means of (1.12), while the function $g(x)$ corresponds to the condition of decreasing function $\Phi(z)$ as $z \to \infty$.

An especially interesting case in the Riemann problem (precisely that which is examined in Sects.2,4—7) is when $\omega = 2\pi$, the coefficient of the problem is a negative number, $G(t) = -G$, $G > 0$, L_{kn} is a set of segments of the real axis $a_{kn}b_{kn}$, $a_{01} < b_{01} < \ldots < b_{0N}$. In this case, we have $\varkappa_n = 1$, $\varkappa = N$ /1/ in the broadest class of functions $\Phi_0(z)$ which are unbounded and integrable near all the endpoints $a_n \equiv a_{0n}$ and $b_n \equiv b_{0n}$. Hence and from (1.9), there follows

$$\Gamma(z) = \frac{1}{2\pi}\int_{L_0} \frac{\ln(-G)e^{it}\,dt}{e^{it} - e^{iz}} = (1/2 - i\gamma)\sum_{n=1}^{N} \ln\frac{e^{iz} - e^{ib_n}}{e^{iz} - e^{ia_n}} \tag{1.14}$$

$$X(z) = \prod_{n=1}^{N}(e^{iz} - e^{ia_n})^{-1/2+i\gamma}(e^{iz} - e^{ib_n})^{-1/2-i\gamma}, \quad \gamma = \frac{\ln G}{2\pi}$$

Of great interest for hydromechanics problems /3/ is the case in which the function $\Phi_0(z)$ is integrable at the points a_n and is bounded at b_n. Then $\varkappa = 0$, according to (1.9)

$$X(z) = \prod_{n=1}^{N}(e^{iz} - e^{ia_n})^{-1/2+i\gamma}(e^{iz} - e^{ib_n})^{1/2-i\gamma} \tag{1.15}$$

The singularities of the function $\Phi_0(z)$ on the contour L_0 are generally conserved even at the appropriate points L_k in the solution $\Phi(z)$. However, in the class of functions decreasing at infinity the inhomogeneous Riemann problem (1.1) also admits of a solution bounded at all points b_{kn}, and the points a_{kn} at some S. As usuall, this will be possible when selecting the function $X(z)$ in the form (1.15) if the function $g(x)$ satisfies S additional integral conditions. Following the known approach (/1/, Sect.44), for an arbitrary contour L_0, the function $X(z)$ can be represented in the most general form

$$X(z) = e^{\Gamma(z)}(e^{\beta z} + 1)^{-\varkappa} \tag{1.16}$$

Excluding (1.13), the solutions constructed are not strict since the passage from problem (1.1) to (1.6) is made formally.

Two paths can apparently be chosen to give it a foundation. The first is associated with utilization of some analog of the Weierstrass theorem about the analyticity of the sum of a series comprised of analytic functions; the conditions of the theorem itself, single-valuedness of the domain $\Omega_0 \setminus L_0$ and unform convergence in z in $\Omega_0 \setminus L_0$, are not satisfied here. The second path is based on uniqueness theorems. Utilization of analyticity of the transform $\Phi_*(z)$ in deriving (1.9) results generally in narrowing the class of allowable solutions of the problem (1.1). However, if this problem has a unique solution, or if the uniqueness theorem is proved for the boundary value problem of mathematical physics akin to (1.1), then the solution of (1.10) will be general in the first case, and sufficient to solve the appropriate physical problem in the second. Uniqueness of the solution of elasticity theory problems for translationally symmetric domains can be proved by the traditional Kirchhoff method by considering the system of partial domains $\Omega_{-N}\cup\Omega_{-N+1}\cdots\cup\Omega_N$ with a growing number of $2N+1$ cells and giving the specific behavior of the solution at infinity. A new approach to this problem is developed in /4/.

In formulating the problem (1.1), and also in the subsequent boundary value problems and examples, it is assumed that their solution decreases at infinity. Solutions different from zero at infinity can be constructed by adding the purely periodic problems considered in /1/ and the work on elasticity theory mentioned below, to the solutions obtained.

2. Pressure of a system of stamps on an elastic half-plane under total adhesion conditions. As an application we examine the problem of the pressure on an elastic half-plane $y\leqslant 0$, which is periodic with period 2π, for a system of stamps loaded arbitrarily and adhering completely to the half-plane boundary $y=0$ at a set of L segments $L_{kn}:[a_n+2k\pi,\ b_n+2k\pi]$, $n=1,\ldots,N$, $k=0,\pm1,\ldots$. The boundary conditions have the form

$$(u+iv)(x-i0)=q(x)+r(x),\ x\in L \qquad (2.1)$$
$$(\sigma_y-i\tau_{xy})(x-i0)=0,\ x\in L'$$

where L' is the continuation of L to the real axis, the function $q(x)=q_1(x)+iq_2(x)$, belonging to the class H_* determines the adhesion condition and the shape of the stamps, the piecewise-constant function $r(x)=r_{kn}^1+ir_{kn}^2$, $x\in L_{kn}$ determines the nature of the connectedness of the stamps.

The problem (2.1) whose solution we seek in the Muskhelishvili form

$$(\sigma_y-i\tau_{xy})(z)=\Phi_1(z)-\Phi_1(\bar z)+(z-\bar z)\overline{\Phi_1'(z)} \qquad (2.2)$$
$$2\mu(u'+iv')(z)=(3-4\nu)\Phi_1(z)+\Phi_1(\bar z)-(z-\bar z)\overline{\Phi_1'(z)}$$

where the prime denotes the derivative with respect to x, ν is the Poisson's ratio, μ is the shear modulus, is reduced /5/ to the Riemann problem (1.1) for the function $\Phi(z)=\Phi_1(z)$, where $G(x)=4\nu-3$, $g(x)=2\mu q'(x)$.

Let the function $q(x)$ be representable as the sum of functions belonging to H_* and H_M. Then the solution is written, respectively, as the superposition of the solutions (1.10), (1.14) and (1.13), (1.14). There remains to find the coefficients C_n of the polynomials $P_{N-1}(e^{iz})$.

We consider the two extreme cases: a) all the stamps are rigidly interconnected, b) all the stamps are displaced independently without rotation, under the action of normal Y_{kn} and tangential X_{kn} applied forces, whose transforms exist in the sense of (1.3) or (1.12). If the system of forces X_{kn}, Y_{kn} is periodic with period $2\pi M$, then it should be self-equilibrated in this period. Conditions (2.1) for the displacements

$$(u+iv)(a_{kn}-i0)=\int_{b_{kn}}^{a_{k,n+1}}(u'+iv')(t)\,dt+q(b_{kn})+r_{kn} \qquad (2.3)$$

should be satisfied in both problems a) and b). According to (2.2), we have

$$(u'+iv')(x)=2\mu^{-1}(1-\nu)\Phi(x),\ x\in L'$$

Hence, and from (2.1) and (2.3), it follows $(k=0,\pm1,\ldots;\ n=1,\ldots,N)$

$$q(a_{k,n+1})+r_{k,n+1}=2\mu^{-1}(1-\nu)\int_{b_n}^{a_{n+1}}\Phi(t+2\pi k)\,dt+q(b_{kn})+r_{kn} \qquad (2.4)$$

Applying the transformation (1.3) or (1.12) to this system, we obtain

$$q_*(a_{n+1})+r_{*,n+1}=2\mu^{-1}(1-\nu)\int_{b_n}^{a_{n+1}}\Phi_*(t)\,dt+q_*(b_n)-r_{*n} \qquad (2.5)$$

where $n=1,\ldots,N$, the function $\Phi_*(t)$ is determined from (1.7), (1.9), (1.14), according to (1.5) and (1.12) for $n=N$ the following notation and equalities must be used

$$a_{N+1}=a_1+2\pi,\quad r_{*,N+}=e^{i\varphi}r_{*1},\quad q_*(a_{N+1})=e^{i\varphi}q_*(a_1),\quad \Phi_*(t)=e^{i\varphi}\Phi_*(t-2\pi),\quad t\in[\pi,a_1+2\pi]$$

By definition $r_{kn} = r = \text{const}$ in problem a), hence the coefficients C_n are found from the system of N equations (2.5) for $r_{*n} = 0,\ n = 1, \ldots, N$.

The C_n in problem b) are determined by the equilibrium conditions

$$\int_{a_{kn}}^{b_{kn}} (\sigma_y - i\tau_{xy})(t)\,dt = Y_{kn} - iX_{kn}\ (n = 1, \ldots, N,\ k = 0, \pm 1, \ldots)$$

which after transformation by (1.3) or (1.12) and utilization of the formula

$$(\sigma_y - i\tau_{xy})(x) = (3 - 4\nu)^{-1}[2\mu q'(x) - 4(1 - \nu)\Phi^+(x)],\ x \in L$$

go over into the system of N equations

$$\frac{4(1-\nu)}{4\nu-3}\int_{a_n}^{b_n} \Phi_*^+(t)\,dt + \frac{2\mu}{3-4\nu}[q_*(b_n) - q_*(a_n)] = Y_{*n} - iX_{*n}$$

The constants r_{kn} are found from the system (2.5) by the inversion formula or by the recursion formulas (2.4), where $r_{01} = 0$. The periodic problem (2.1) is solved in /6/.

3. The Hilbert problem for a half-plane. Find the function $F(z)$ decreasing at infinity, which is analytic in the half-plane $y > 0$ and continuable continuously at its boundary $y = 0$ except, perhaps, at given points $x = d_n + 2\pi k,\ y = 0$ ($|d_n| \leqslant \pi,\ n = 1, \ldots, 2N;$ $k = 0, \pm 1, \ldots$) at which this function should be integrable by the boundary condition

$$a(x)\,\mathrm{Re}\,F(x) + b(x)\,\mathrm{Im}\,F(x) = c(x),\ x \in (-\infty, \infty) \tag{3.1}$$

Here $a(x)$, $b(x)$ are real functions satisfying the condition $H_1(\omega = 2\pi)$ with the exception of points $x = d_n$, where discontinuities of the first kind are allowable, $a^2(x) + b^2(x) \neq 0$ for $x \in [-\pi, \pi]$; and $c(x)$ is a real function belonging to the class H_*.

According to /7/, the solution of the problem (3.1) has the form

$$F(z) = {}^1/_2\,[\Phi(z) + \bar{\Phi}(z)] \tag{3.2}$$

where $\Phi(z)$ is the solution of the Riemann problem (1.1) on the whole line $y = 0$, where

$$G(x) = -\frac{a(x) + ib(x)}{a(x) - ib(x)},\quad g(x) = \frac{2c(x)}{a(x) - ib(x)} \tag{3.3}$$

Therefore, the solution of the problem (3.1) is expressed by (3.2) and (1.10) or (1.13), where L_0 is the segment $[-\pi, \pi]$, the function $X(z)$ can be calculated at the points d_n by means of (1.9) or (1.16) depending on the behavior of the function $G(x)$, and we have according to (1.9) and (3.3)

$$\Gamma(z) = -\frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{2\arg[a(t) + ib(t)] + \pi}{e^{it} - e^{iz}}\,e^{it}\,dt \tag{3.4}$$

Analogously, by using the results in /8/, the quasiperiodic Hilbert problem for a plane with slits on the real axis can be solved.

4. Pressure of a system of stamps on an elastic half-plane under limit friction conditions. We examine the quasiperiodic problem of the pressure on an elastic half-plane $y < 0$ of a system of stamps, exactly as in Sect.2, but under limit friction conditions on the line of contact. In the notation of Sect.2, the boundary conditions of this problem have the form

$$v(x) = q(x) + r(x),\ (\tau_{xy} + \rho\sigma_y)(x) = 0,\ x \in L \tag{4.1}$$
$$(\sigma_y - i\tau_{xy})(x) = 0,\ x \in L'$$

Here $|\rho|$ is the friction coefficient, the real function $q(x)$ governing the shape of the stamp belongs to the class H_* or H_M, $r(x) = r_{kn}$, $x \in L_{kn}$, and r_{kn} are real constants.

We construct the solution in the form (2.2). According to /5/, the function $F(z) = (i + \rho)\,\Phi_1(z)$ is a solution of the Hilbert problem (3.1), where

$$a(x) = 4(1 - \nu),\quad b(x) = 2\rho(1 - 2\nu),\quad c(x) = -2\mu(1 + \rho^2)q'(x),\ x \in L \tag{4.2}$$
$$a(x) = 0,\ b(x) = 1,\ c(x) = 0,\ x \in L'$$

According to (3.3) and (1.14), we obtain for the appropriate Riemann problem

$$X(z)=\prod_{n=1}^{N}\left(e^{iz}-e^{ia_n}\right)^{-1/2-\theta}\left(e^{iz}-e^{ib_n}\right)^{-1/2+\theta} \tag{4.3}$$
$$\theta=\pi^{-1}\operatorname{arctg}\left[\rho(1-2\nu)(2-2\nu)^{-1}\right]$$

The function $\Phi(z)$ is expressed by (1.10) (or (1.11), (1.13)), (3.3), (4.2), (4.3), the function $F(z)$ by (3.2), $\Phi_1(z)=(i+\rho)^{-1}F(z)$.

We find the coefficients C_n. From condition (4.1) we obtain

$$\int_{b_{kn}}^{a_{k,n+1}} v'(t)\,dt=q(a_{k,n+1})-q(b_{kn})+r_{k,n+1}-r_{kn};\quad n=1,\ldots,N \tag{4.4}$$

Taking into account that $\bar{\Phi}(x)=\overline{\Phi(x)}$ outside the stamp, we obtain from (2.2)

$$v'(x)=-(1-\nu)\mu^{-1}(1+\rho^2)^{-1}\left[\Phi(x)+\overline{\Phi(x)}\right] \tag{4.5}$$

Substituting (4.5) into (4.4), we obtain N conditions for each $k=0,\pm1,\ldots$

$$\frac{2(1-\nu)}{\mu(1+\rho^2)}\int_{b_n}^{a_{n+1}}\operatorname{Re}\Phi(t+2\pi k)\,dt=q(b_{kn})-q(a_{k,n+1})+r_{kn}-r_{k,n+1} \tag{4.6}$$

that generate a system of N integral equations in $C_n(\varphi)$ after transformation with respect to k. However, it can be noted that conditions (4.6) will be satisfied if $\operatorname{Re}\Phi(t+2\pi k)$ therein is replaced by the function $\Phi(t+2\pi k)$ itself. Later, applying the Fourier transform, we obtain a system of N algebraic equations with the unknowns $C_n(\varphi)$

$$\frac{2(1-\nu)}{\mu(1+\rho^2)}\int_{b_n}^{a_{n+1}}\Phi_*(t)\,dt=q_*(b_n)-q_*(a_{n+1})+r_{*n}-r_{*,n+1}; \tag{4.7}$$
$$n=1,\ldots,N$$

Analogously to Sect.2 in problem a), here we must set $r_{*n}=0$, the magnitudes of the principal vectors Y_{kn} are found from the equilibrium conditions

$$\int_{a_n}^{b_n}\sigma_y(x+2\pi k)\,dx=Y_{kn};\quad n=1,\ldots,N,\quad k=0,\pm1,\ldots \tag{4.8}$$

The equalities (4.7) in problem b) serve to calculate the transform of the relative vertical displacements of the stamps r_{*n}. The coefficients C_n are determined by the conditions (4.8). By virtue of (2.2) and (4.1), we have in the interval (a_{kn}, b_{kn})

$$\sigma_y(x)=-(1+\rho^2)^{-1}\operatorname{Im}\left[\Phi^+(x)-\Phi^-(x)\right] \tag{4.9}$$

Substituting (4.9) into (4.8), and using the same method as in deriving (4.7), we obtain the system $(n=1,\ldots,N)$

$$\frac{i}{1+\rho^2}\int_{a_n}^{b_n}\left[\Phi_*^{+}(t)-\Phi_*^{-}(t)\right]dt=Y_{*n} \tag{4.10}$$

Despite the fact that not necessary but sufficient conditions (4.7) and (4.10) are used in the solutions constructed, they will be general if they are unique under conditions of damping at infinity. It should be noted that the Kirchhoff uniqueness theorem does not include the limit friction conditions by eliminating the case $\rho=0$. For $\rho=0$ the formulas (4.3)– (4.10) are simplified somewhat (for instance, $\theta=0$) and correspond to the case of absence of contact friction. Periodic problems for $\rho=0$ are solved in /9/, and for $\rho\neq0$ in paper /10/.

5. Mixed boundary value problem of analytic function theory for a half-plane. Find the function $F(z)$ analytic for $y>0$ by means of the boundary condition

$$\operatorname{Re}F(x)=q(x),\ x\in L;\ \operatorname{Im}F(x)=h(x),\quad x\in L' \tag{5.1}$$

where the functions $q(x)$ and $h(x)$ independently satisfy the conditions H_* or H_M.

This problem is a particular case of the Hilbert problem (3.1) for $a(x)=1$, $b(x)=0$, $c(x)=q(x)$ on L and $a(x)=0$, $b(x)=1$, $c(x)=h(x)$ on L'. Hence its solution which decreases at infinity and is integrable at the points a_{kn}, b_{kn}, is expressed by (3.2) and (1.10) or (1.13), where $g(x) = 2q(x)$ on L and $g(x) = 2ih(x)$ on L'. In particular, if the solution $F(z)$ bounded at all points b_{nk} is sought, then the function $X(z)$ is evaluated by means of (1.15), where $\gamma = 0$ since $G(x) = -1$ on L, $G(x) = 1$ on L', here $\varkappa = 0$, and there are no arbitrary constants $C_n(\varphi)$. If singularities are allowable at both ends of the segments L by the condition of the problem, then (1.14) must be used, and $\gamma = 0, \varkappa = N$. The formulas mentioned can be considered a generalization of the Keldysh–Sedov formulas /3/.

6. Dirichlet problem for a plane with slits. The following modification of this problem plays a major role in hydromechanics and elasticity theory. Construct a function $\Psi(z)$ analytic and single-valued in a plane z slit in a set of intervals L by means of given values of its real part

$$\operatorname{Re}\Psi^{\pm}(x) = q^{\pm}(x), \quad x \in L, \quad q^{\pm}(x) \in H_M, \quad H_* \tag{6.1}$$

Following /1,7/, we write the solution in the form

$$\Psi(z) = F(z) + \Omega(z) \tag{6.2}$$

where $F(z)$ is the solution of the mixed problem (5.1) for $h(x) = 0$, $q(x) = {}^1/_2\,[q^+(x) + q^-(x)]$, and $\Omega(z)$ is the solution of the problem of the jump

$$\Omega^+(x) - \Omega^-(x) = \omega(x), \; \omega(x) = q^+(x) - q^-(x), \; x \in L \tag{6.3}$$

with the additional condition $\Omega(z) = -\bar{\Omega}(z)$.

According to Sect.5, the function $F(z)$ is expressed by (3.2), (1.10) or (1.13), where

$$g_*(x) = q_*^+(x) + q_*^-(x), \quad x \in (a_n, b_n); \tag{6.4}$$
$$g_*(x) = 0, \quad x \in [b_n, a_{n+1}]$$

For $\gamma = 0$ the function $X(z)$ is expressed by (1.14) or (1.15) to which $\varkappa = N$ or $\varkappa = 0$ corresponds.

By virtue of (1.10), the solution of the problem (6.3) has the form

$$\Omega(z) = {}^1/_2\,[\Phi(z) - \bar{\Phi}(z)] \tag{6.5}$$

$$\Phi(z) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_{L_2} \frac{\omega_*(t)\exp(it + i\alpha z - i\alpha t)}{e^{it} - e^{iz}}\,dt\,d\varphi \tag{6.6}$$

The solution of the Dirichlet problem, as a problem to determine a harmonic function from its values on the edges of slits, is constructed on the basis of the solution obtained by the methods in /7/ and /11/, As is known, this function is generally the real part of a multivalued analytic function.

7. Deformation of a composite elastic plane weakened by a system of closed slits. Let an elastic plane z be glued from the half-planes $y \geqslant 0$ and $y \leqslant 0$ with different elastic characteristics, and weakened on the interface of the materials by a system of L arbitrarily loaded closed slits. This quasiperiodic problem for a homogeneous plane and the corresponding periodic problem for a composite plane are solved in /2/ and /12/.

We write down the boundary conditions on L $(h^{\pm}(x) \in H_*, H_M)$

$$\tau_{xy}(x \pm i0) = h^{\pm}(x), \; v(x + i0) = v(x - i0) \tag{7.1}$$
$$\sigma_y(x + i0) = \sigma_y(x - i0)$$

Considering total adhesion of the half-planes to hold on L', we construct the solution of the problem satisfying this condition in the A.A. Khrapkov form /13/

$$2\mu_j(u + iv)'(z) = c_j[\chi_j K(z) - (z - \bar{z})\overline{K'(z)}] - \delta_j K(\bar{z}) + \tag{7.2}$$
$$c_{j+2}[\chi_j M(z) + M(\bar{z}) - (z - \bar{z})\overline{M'(z)}]$$
$$\chi_j = 3 - 4\nu_j, \; \delta_1 = c_2, \; \delta_2 = c_1, \; c_1 = (\chi_1 + m)^{-1}$$
$$c_2 = (1 + m\chi_2)^{-1}, \quad c_3 = m(1 + \chi_2), \quad c_4 = 1 + \chi_1, \quad m = \mu_1\mu_2^{-1}$$

where the subscript $j = 1$ $(j = 2)$ denotes the elastic characteristics of the half-planes $y \geqslant 0$ $(y \leqslant 0)$.

Substituting (7.2) into (7.1), we obtain the following boundary value problems for the functions $M(z)$ and $\Psi(z)$

$$M^+(x) - M^-(x) = ip(x), \; x \in L \tag{7.3}$$

$$\begin{aligned}
&\operatorname{Re}\Psi^{\pm}(x) = q^{\pm}(x),\ \ x \in L;\ \ \Psi(z) = -i(c_1 + c_2)K(z)\\
&p(x) = -\Delta\,[h^+(x) - h^-(x)],\ \ \Delta = (c_3 + c_4)^{-1}\\
&q^+(x) = c_2 r(x) + q(x),\ \ q^-(x) = -c_1 r(x) + q(x)\\
&q(x) = -\Delta\,[c_4 h^+(x) + c_3 h^-(x)],\ \ r(x) = m\Delta(\chi_1\chi_2 - 1)\times[h^+(x) - h^-(x)]
\end{aligned} \tag{7.4}$$

The solution of the problem of the jump (7.3) has the form

$$M(z) = \frac{i}{4\pi^2}\int_0^{2\pi}\int_{L_*}\frac{p_*(t)\exp(it + i\alpha z - i\alpha t)}{e^{it} - e^{iz}}\,dt\,d\varphi \tag{7.5}$$

The solution of the Dirichlet problem (7.4) has the form (6.2), where the function $\Omega(z)$ is determined by (6.5), (6.6), (6.3), the function $F(z)$ by (3.2), (1.10), (1.13), the functions $g_*(x)$ and $X(z)$ have the form (6.4) and (1.14) for $\gamma = 0,\ \varkappa = N$.

We find the coefficients C_n. We write the conditions for uniqueness of the displacements $u(z)$ during traversal of the slits

$$\int_{a_{kn}}^{b_{kn}}[u'(t + i0) - u'(t - i0)]\,dt = 0;\ \ n = 1, \ldots, N;\ \ k = 0, \pm 1, \ldots$$

Substituting (7.2) here, we obtain for the same k and n

$$\int_{a_{kn}}^{b_{kn}}\operatorname{Re}\{m(\chi_1\chi_2 - 1)[M^+(t) - M^-(t)] + K^+(t) - K^-(t)\}\,dt = 0 \tag{7.6}$$

Since the function $M(z)$ has a pure imaginary jump on L according to (7.3), and the coefficients χ_j, μ_j are real, it follows from (7.6)

$$\int_{a_{kn}}^{b_{kn}}\operatorname{Re}[K^+(t) - K^-(t)]\,dt = 0$$

We go over to the functions $\Phi(z)$ and $\Omega(z)$ in this equation by means of (7.4), (6.2) and (3.2). Taking into account that the imaginary part of the jump in the function $\Omega(z)$ is zero on L because of (6.3), we obtain

$$\int_{a_{kn}}^{b_{kn}}\operatorname{Im}[\Phi^+(t) - \Phi^-(t)]\,dt = 0 \tag{7.7}$$

To satisfy this condition it is sufficient to write the jump itself in the function $\Phi(z)$ instead of the imaginary part of the jump. By applying the transformation (1.3) or (1.12) to the new condition, we obtain a system of N equations with the unknowns $C_n(\varphi)$

$$\int_{a_n}^{b_n}[\Phi_*^+(t) - \Phi_*^-(t)]\,dt = 0,\ \ n = 1, \ldots, N \tag{7.8}$$

For $h^{\pm}(x) \equiv 0$, this system and the problem (7.1) have only a trivial solution. Going from (7.7) over to (7.8) and from (1.1) over to (1.6), a foundation can here be given by using the Kirchhoff uniqueness theorem.

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